Something different on arrays

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Consider the following six by six array (Figure 1):

2	5	12	11	14	17
10	13	20	19	22	25
34	37	44	43	46	49
27	30	37	36	39	42
18	21	28	27	30	33
42	45	52	51	54	57

Figure 1

If asked to pick six numbers from this array, no two of which are in the same row or column, you could proceed as follows:

1. Choose any number from the array, and then cross out all remaining numbers from the row and column in which it is lies (the numbers chosen in this example appear in bold enlarged print in Figures 2–4).
2. Choose a second number from the numbers not yet eliminated and then strike out all remaining numbers from the row and column in which it appears (see Figure 3).
3. Similarly choose three additional numbers (5, 46 and 42), thus arriving at Figure 4.
4. When "choosing" the sixth and final number, there is hardly any choice.

2	5	12	~~11~~	14	17
~~10~~	~~13~~	~~20~~	**19**	~~22~~	~~25~~
34	37	44	~~43~~	46	49
27	30	37	~~36~~	39	42
18	21	28	~~27~~	30	33
42	45	52	~~51~~	54	57

Figure 2

2	5	~~12~~	~~11~~	14	17
~~10~~	~~13~~	~~20~~	**19**	~~22~~	~~25~~
34	37	~~44~~	~~43~~	46	49
~~27~~	~~30~~	**37**	~~36~~	~~39~~	~~42~~
18	21	~~28~~	~~27~~	30	33
42	45	~~52~~	~~51~~	54	57

Figure 3

~~2~~	**5**	~~12~~	~~11~~	~~14~~	~~17~~
~~10~~	~~13~~	~~20~~	**19**	~~22~~	~~25~~
~~34~~	~~37~~	~~44~~	~~43~~	**46**	~~49~~
~~27~~	~~30~~	**37**	~~36~~	~~39~~	~~42~~
~~18~~	~~21~~	~~28~~	~~27~~	~~30~~	***33***
42	~~45~~	~~52~~	~~51~~	~~54~~	~~57~~

Figure 4

Having already chosen numbers in the first, second, third, fourth, and sixth rows as well as in the first, second, third, fourth and fifth columns, we must select a number from the remaining row (row five) as well as from the column yet to be chosen (column six). This leaves us with the number 33 found in row five and column six of Figure 4.

5. Add the six numbers chosen and write down the sum of those numbers:

$$5 + 19 + 33 + 37 + 46 + 42 = 182$$

At this point we invite the reader to choose a different set of six numbers from the array above using the procedure outlined in steps 1–4 and calculate their sum. What is remarkable is that the same total, 182, will be obtained!

We intend to:

- explain why this is so;
- focus on arrays resembling the given illustration, and determine the conditions that assure us that the sum of any six numbers obtained in the manner outlined above will be invariant;
- show how one may predetermine the sum of any six numbers picked by the procedure described in steps 1–4 above.

In this paper, we utilise the following definition:

Definition 1: A k-translation of a sequence

Let the elements of a sequence be a_1, a_2, a_3, … a_n; the k-translation of that sequence is the sequence $a_1 + k$, $a_2 + k$, $a_3 + k$, … $a_n + k$ for any non-zero number k.

Example 1

Were we to consider the numbers in the first row of the array in Figure 1 in the order of their appearance as a sequence:

$$2, 5, 12, 11, 14, 17$$

then the second row of the array, when viewed as a number sequence, is an 8-translation of the row-one sequence:

$$2 + 8, 5 + 8, 12 + 8, 11 + 8, 14 + 8, 17 + 8$$

or

$$10, 13, 20, 19, 22, 25$$

The arithmetic progression translation property

We assume without proof the following property of all arithmetic progressions: the sequence a_1, a_2, a_3, … a_n is an arithmetic progression if and only for any constant k, its k-translation $a_1 + k$, $a_2 + k$, $a_3 + k$, … $a_n + k$ is also an arithmetic progression.

This means that increasing (or decreasing) all numbers in an arithmetic progression by a constant yields another arithmetic progression. We refer to this property henceforth as the APT property.

A generalised array satisfying the invariance requirement

We now show that the generalised six-by-six array in Figure 5 meets the conditions needed to yield the same sum of any six numbers taken from the array, provided that these numbers include representatives of all rows and columns.

We define $a_1 = 0$ and $b_1 = 0$ in this array, but include "+ a_1" and "+ b_1" in all the elements of column one and row one respectively to preserve uniformity. As a result, all array elements assume the form $a + a_r + b_s$, r and s being integers between 1 and 6 (with the value of subscript s not dependent on that of r). We note that the element in the array corresponding to $a + a_r + b_s$ lies in row s and column r.

$a + a_1 + b_1$	$a + a_2 + b_1$	$a + a_3 + b_1$	$a + a_4 + b_1$	$a + a_5 + b_1$	$a + a_6 + b_1$
$a + a_1 + b_2$	$a + a_2 + b_2$	$a + a_3 + b_2$	$a + a_4 + b_2$	$a + a_5 + b_2$	$a + a_6 + b_2$
$a + a_1 + b_3$	$a + a_2 + b_3$	$a + a_3 + b_3$	$a + a_4 + b_3$	$a + a_5 + b_3$	$a + a_6 + b_3$
$a + a_1 + b_4$	$a + a_2 + b_4$	$a + a_3 + b_4$	$a + a_4 + b_4$	$a + a_5 + b_4$	$a + a_6 + b_4$
$a + a_1 + b_5$	$a + a_2 + b_5$	$a + a_3 + b_5$	$a + a_4 + b_5$	$a + a_5 + b_5$	$a + a_6 + b_5$
$a + a_1 + b_6$	$a + a_2 + b_6$	$a + a_3 + b_6$	$a + a_4 + b_6$	$a + a_5 + b_6$	$a + a_6 + b_6$

Figure 5

Since b_1 is zero by definition, $a + a_r + b_2 = (a + a_r + b_1) + b_2$ for each a_r, $r = 1, 2, 3, 4, 5$ or 6. This means that the sequence in row two of Figure 5 is a b_2-translation of the sequence of numbers in row one. Similarly, the sequences in rows three, four, five and six are b_3-, b_4-, b_5- and b_6-translations respectively of the row-one sequence.

We now intend to show that requiring all successive rows in a six by six array be translations of the first row sequence ensures that the sum of the six numbers chosen from the Figure 5 array in the manner described above will always be invariant.

Theorem 1

If a six-by-six array assumes the form of the generalised six-by-six array of Figure 5, then no matter which six numbers are chosen — so long as no pair of them lie in the same row or column — their sum is invariant.

Proof

1. Upon examination of the generalised array in Figure 5, we find that for each a_r and b_s, $r = 1, 2, 3, 4, 5$ or 6 and $s = 2, 3, 4, 5$ or 6:

$$a + a_r + b_s = (a + a_r + b_1) + b_s \text{ (since } b_1 = 0 \text{ by definition)}.$$

This means that for each s, $s = 2, 3, 4, 5$ or 6, the sth row of the array,

when viewed as a number sequence, is a b_s-translation of the first row. We now show that this property ensures that the sum of the six numbers picked, no two of which lie in the same row or column is invariant.

2. Next, we note the following:
   (a) All elements in every row s, $s = 1, 2, 3, 4, 5$ or 6 of the generalised array contain the addend b_s. For example, all elements of row 3 contain the addend b_3. On the other hand, b_3 does not appear as an addend in the expression representing any element of a different row.
   (b) Similarly, all elements in every column, r, $r = 1, 2, 3, 4, 5$ or 6 contain an addend a_r. For example, all elements of column 2 above contain the addend a_2, but a_2 does not appear in the expression representing an element of any other column.
   (c) From the above we conclude that for every integral value of m and n between 1 and 6, there is a one-to-one correspondence between the expression $a + a_m + b_n$ and the element situated in row n and column m of the array.

3. Suppose we have chosen six elements from the array through the method described above, no two of which lie in the same row or column. The sum of the expressions representing these elements is:

$$(a + a_g + b_l) + (a + a_i + b_j) + (a + a_m + b_n) + (\quad) + (\quad) + (a + a_f + b_t)$$

   (a) We note that each of the six expressions to be summed contains a. This contributes $6 \cdot a$ to the overall sum.
   (b) Furthermore, since each of the elements comes from a different column, each one of the six expressions will contain a different a_r. Consequently when adding these six expressions — regardless of the original choice of elements — and following appropriate rearrangement and regrouping of these terms, the sum of the six numbers picked from the array will include a component $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$.
   (c) Similarly, since the six numbers to be added have been chosen from different rows, each of the six expressions representing them will contain a different b_s. After rearranging and regrouping these terms, we find that their constituent sum will be $b_1 + b_2 + b_3 + b_4 + b_5 + b_6$, irrespective of the array numbers chosen randomly.
   (d) It follows from the above that the sum of the expressions representing the selected six numbers will always be

$$6 \cdot a + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)$$

This means that the sum of any six numbers of the array chosen from different rows and columns never changes, so long as the sequence of numbers in rows two through six are translations of the number sequence in row one.

As for predicting the invariant six-number *sum* for a given array, one can calculate the sum of the numbers lying along either array diagonal *before* the selection of the six numbers. Since the six numbers along either diagonal belong to different rows and different columns they are valid six-number selections and their sum provides the invariant result.

Thus, in the original illustration, one could have predicted the sum of the six arbitrary numbers to be chosen from different rows and columns by summing the numbers along either diagonal of Figure 1:

$$2 + 13 + 44 + 36 + 30 + 57 \text{ or } 17 + 22 + 43 + 37 + 21 + 42,$$

both equalling 182. Either calculation easily provides the expected result when the six numbers are chosen randomly as described above.

A special case

In the event that the sequences $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ are both arithmetic progressions[1], one can readily show that the sum of the six elements chosen as described above is

$$6 \cdot a + 3 \cdot (a_6 + b_6)$$

This follows from the fact that the sum of an arithmetic progression $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$ equals

$$6 \cdot \frac{(a_1 + a_6)}{2}$$

or $3 \cdot a_6$ (since a_1 equals zero by definition).

In like manner it follows that the sum $b_1 + b_2 + b_3 + b_4 + b_5 + b_6$ equals $3 \cdot b_6$. Thus we conclude that whenever both the first column and the first row are arithmetic progressions[2], the "fixed" sum

$$6 \cdot a + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)$$

can be replaced by $6 \cdot a + 3 \cdot (a_6 + b_6)$.

We are able to replace this last result by a more useful formula as follows: Since a_1 and b_1 are both zero by definition, then

$$6 \cdot a + 3 \cdot (a_6 + b_6) = 3 \cdot (a + a_1 + b_1) + 3 \cdot (a + a_6 + b_6) = 3 \cdot [(a + a_1 + b_1) + (a + a_6 + b_6)]$$

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1. If the sequences $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ are both arithmetic progressions, then the APT property ensures that the sequences in the first row and column of Figure 5 will also be arithmetic progressions (both are translations of sequences assumed to be arithmetic progressions).
2. See (1).

We prefer to restate the last expression as follows:

$$3\cdot\left[\left(a+a_1+b_1\right)+\left(a+a_6+b_6\right)\right]=6\cdot\frac{\left(a+a_1+b_1\right)+\left(a\ +\ a_6+b_6\right)}{2}$$

Conclusion in the special case

If in a six by six array in which all rows are translations of the first row, and the numbers in both the first row and column form arithmetic progressions, then the sum of any six numbers of the array, no two of which are in the same row or column, is six times the arithmetic mean of the numbers in the upper left and lower right corners of the array.

Verification of the preceding formula

Example 2

Show that the formula just derived applies to the array in Figure 6, and utilise that formula to determine the sum of any sextet of numbers drawn from that array, no two lying in the same row or column.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Figure 6

Solution

1. Both the first row and the first column are arithmetic progressions: 1, 2, 3, 4, 5, 6 and 1, 7, 13, 19, 25, 31.
2. In addition, rows 2, 3, 4, 5 and 6 are a 6-translation, a 12-translation, an 18-translation, a 24-translation and a 30-translation respectively of the sequence in row 1.
3. In this example, 1 and 36 are the two relevant corner elements.
4. It follows from our conclusion above that the sum of any sextet drawn from the given array, no two of which lie in the same row or column equals $$6\cdot\frac{1+36}{2}=3\cdot 37=111$$

Note: Were we to sum the numbers along either diagonal, we would obtain the same result:

$$1 + 8 + 15 + 22 + 29 + 36 = 6 + 11 + 16 + 21 + 26 + 31 = 111$$

Multiplicatively invariant arrays

We have produced the array in Figure 7 by systematically altering symbols appearing in Figure 5. For each addition sign in Figure 5 we have substituted a multiplication sign. Subscripts were left unchanged, and each small letter replaced by the corresponding large letter.

$A \cdot A_1 \cdot B_1$	$A \cdot A_2 \cdot B_1$	$A \cdot A_3 \cdot B_1$	$A \cdot A_4 \cdot B_1$	$A \cdot A_5 \cdot B_1$	$A \cdot A_6 \cdot B_1$
$A \cdot A_1 \cdot B_2$	$A \cdot A_2 \cdot B_2$	$A \cdot A_3 \cdot B_2$	$A \cdot A_4 \cdot B_2$	$A \cdot A_5 \cdot B_2$	$A \cdot A_6 \cdot B_2$
$A \cdot A_1 \cdot B_3$	$A \cdot A_2 \cdot B_3$	$A \cdot A_3 \cdot B_3$	$A \cdot A_4 \cdot B_3$	$A \cdot A_5 \cdot B_3$	$A \cdot A_6 \cdot B_3$
$A \cdot A_1 \cdot B_4$	$A \cdot A_2 \cdot B_4$	$A \cdot A_3 \cdot B_4$	$A \cdot A_4 \cdot B_4$	$A \cdot A_5 \cdot B_4$	$A \cdot A_6 \cdot B_4$
$A \cdot A_1 \cdot B_5$	$A \cdot A_2 \cdot B_5$	$A \cdot A_3 \cdot B_5$	$A \cdot A_4 \cdot B_5$	$A \cdot A_5 \cdot B_5$	$A \cdot A_6 \cdot B_5$
$A \cdot A_1 \cdot B_6$	$A \cdot A_2 \cdot B_6$	$A \cdot A_3 \cdot B_6$	$A \cdot A_4 \cdot B_6$	$A \cdot A_5 \cdot B_6$	$A \cdot A_6 \cdot B_6$

Figure 7

The element in the sth row and rth column of Figure 7 is $A{\cdot}A_r{\cdot}B_s$ whereas the corresponding element in Figure 5 was $a + a_r + b_s$. Examining Figure 7 we find that A_r appears as a factor in all elements of column r, $r = 1, 2, \ldots 6$ whereas B_s appears as a factor in all elements of row s, $s = 1, 2, \ldots 6$.

We have established that Figure 5 represents the general form of a 6 by 6 additively invariant array (AIA). We intend to convince the reader that the general array in Figure 7 provides 6-element products that are invariant so long as no two of the six elements are chosen from the same row or column. We refer to such arrays as multiplicatively invariant arrays (MIA).

We surmise that products of six elements of the array in Figure 7 lying in different rows and columns are multiplicatively invariant from the following considerations:

1. Each of the 6 elements in such products will contribute a factor of A.
2. Each of the 6 elements in any "appropriate" product contributes a factor A_r and a factor of B_s where both r and s must take on the values 1, 2, 3, 4, 5 and 6 precisely once. For example, A_3 cannot appear twice among the 6 element factors chosen because this would mean that two of the elements were taken from column 3, and we have required that no two elements come from the same column. Similarly B_5 could not appear more than once among the 6 arbitrary element factors chosen, because this would mean we had chosen two different elements from row 5 of the array.
3. It follows from the preceding that the product obtained by multiplying 6 arbitrary elements, no two of which lie in the same row or column — after appropriate regrouping and reordering of all the factors — will always be
$$(A{\cdot}A{\cdot}A{\cdot}A{\cdot}A{\cdot}A)\cdot(A_1{\cdot}A_2{\cdot}A_3{\cdot}A_4{\cdot}A_5{\cdot}A_6)\cdot(B_1{\cdot}B_2{\cdot}B_3{\cdot}B_4{\cdot}B_5{\cdot}B_6),$$
or $$A^6{\cdot}(A_1{\cdot}A_2{\cdot}A_3{\cdot}A_4{\cdot}A_5{\cdot}A_6)\cdot(B_1{\cdot}B_2{\cdot}B_3{\cdot}B_4{\cdot}B_5{\cdot}B_6).$$
This assures us that the product will never change.

We still need to determine the characteristics of the general array in Figure 7 that guarantee it is an MIA. For this purpose we introduce the following definition.

Definition 2: A k-magnification of a sequence

Let the elements of a sequence be A_1, A_2, A_3, … A_n; the k-magnification of that sequence is the sequence $A_1 \cdot k$, $A_2 \cdot k$, $A_3 \cdot k$, … $A_n \cdot k$ for any non-zero number k.

Example 3

Suppose we are given the sequence: 2, –1, 7, 4, –3, 5. A 3-magnification of that sequence is the sequence 2·3, (–1)·3, 7·3, 4·3, (–3)·3, 5·3 or 6, –3, 21, 12, –9, 15.

We now apply this definition to the array in Figure 7 after assigning $B_1 = 1$.[3] Let us choose as a sequence the first row of that general array: $A \cdot A_1 \cdot B_1$, $A \cdot A_2 \cdot B_1$, $A \cdot A_3 \cdot B_1$, $A \cdot A_4 \cdot B_1$, $A \cdot A_5 \cdot B_1$, $A \cdot A_6 \cdot B_1$. Since we have assigned to B_1 the value 1, we may rewrite the sequence as $A \cdot A_1$, $A \cdot A_2$, $A \cdot A_3$, $A \cdot A_4$, $A \cdot A_5$, $A \cdot A_6$. Then a B_2-magnification of that sequence will be $A \cdot A_1 \cdot B_2$, $A \cdot A_2 \cdot B_2$, $A \cdot A_3 \cdot B_2$, $A \cdot A_4 \cdot B_2$, $A \cdot A_5 \cdot B_2$, $A \cdot A_6 \cdot B_2$. Lo and behold, this is nothing less than the sequence in row 2 of our Figure 7 array! Similarly we find that the arrays in rows 3, 4, 5 and 6 of Figure 7 are B_3-, B_4-, B_5- and B_6-magnifications respectively of the sequence in row 1 of the array, provided we define $B_1 = 1$. From this we surmise the following: if all rows in a six by six array except the first one are magnifications of the sequence of numbers in row 1, then multiplying any 6 numbers from the array, no two lying in the same row or column always yields the same product. We illustrate this result using the array in Figure 8.[4]

1	2	4	8	16	32
3	6	12	24	48	96
9	18	36	72	144	288
27	54	108	216	432	864
81	162	324	648	1296	2592
243	486	972	1944	3888	7776

Figure 8

Example 4

Show that the array in Figure 8 is multiplicatively invariant, and then determine the product to be obtained by multiplying any 6 elements (numbers) of the array, no two of which lie in the same row or column.

Solution

Based on our deductions above, the array is a MIA, being that the sequences in row 2, 3, 4, 5 and 6 are 3-, 9-, 27-, 81- and 243-magnifications respectively of

3. Why B_1 is assigned the value 1 is explained below.
4. The particular choice of numbers in Example 4, though cumbersome, was motivated by the ease with which one can solve the same problem in Example 5 using an improved formula.

the row one sequence. To predetermine the product of any 6 elements of the array, no two lying in the same row or column, it suffices to multiply the numbers along one of the diagonals: $1\cdot 6\cdot 36\cdot 216\cdot 1296\cdot 7776 = 470\,184\,984\,576$. This is so because all the numbers along either diagonal lie in different rows and columns (just as in an AIA, or additively invariant array).

An AIA–MIA Comparison

Let us re-examine the relationship between additively and multiplicatively invariant 6 by 6 arrays. We begin by comparing their features in Table 1.

Table 1

terms/operations in AIA		**terms/operations in MIA**
sum		product
$a + a_r + b_s$ (element in sth row and rth column)		$A\cdot A_r\cdot B_s$ (element in sth row and rth column)
$b_1 = 0$		$B_1 = 1$
$a + a + a + a + a + a = 6\cdot a$	*replaced by*	$A\cdot A\cdot A\cdot A\cdot A\cdot A = A^6$
summed		multiplied
$6\cdot a+(a_1+a_2+a_3+a_4+a_5+a_6)+(b_1+b_2+b_3+b_4+b_5+b_6)$ is the invariant sum		$A^6\cdot(A_1\cdot A_2\cdot A_3\cdot A_4\cdot A_5\cdot A_6)\cdot(B_1\cdot B_2\cdot B_3\cdot B_4\cdot B_5\cdot B_6)$ is the invariant product
translation		magnification
b_3-translation		B_3-magnification

In moving from Figure 5 to Figure 7, we replaced small letters and addition signs by large letters and multiplication signs respectively. Our setting $b_1 = 0$ for AIA but $B_1 = 1$ for MIA is due to these assignments corresponding to the identity elements for addition and multiplication. These substitutions led to the sequences in all rows other than the first one being translations of the first row sequence in an AIA and magnifications of the first row sequence in an MIA. In one case we add the same constant to all terms of a sequence, whereas in the dual system we multiply each term of a sequence by the same constant.

The interrelationships between AIA and MIA are so pronounced that appropriate replacements using Table 1, as well as Table 2 that follows, led the writer to a result for MIA that matches a formula for the special case of AIA presented above. In that situation both sequences a_1, a_2, a_3, a_4, a_5, a_6 and b_1, b_2, b_3, b_4, b_5, b_6 were arithmetic progressions. That meant that there existed constants c and k such that $a_{m+1} = a_m + c$ and $b_{m+1} = b_m + \mathrm{k}$ for $m = 1, 2, \ldots 5$. To find a formula for an MIA comparable to the one discovered in the special case pertaining to an AIA, it would be necessary that there must exist constants C and K such that $A_{m+1} = A_m\cdot C$ and $B_{m+1} = B_m\cdot K$ for $m = 1, 2, \ldots 5$. The latter two conditions are equivalent to requiring that the sequences A_1, A_2, A_3, $\ldots A_6$ and B_1, B_2, B_3, $\ldots B_6$ both be geometric progressions.

Furthermore, for the same reason that $b_1 = 0$ had to correspond to $B_1 = 1$, so did $A_1 = 1$ need to be the condition analogous to $a_1 = 0$ in the special case for AIA. The analogy further required that the product of the terms of a geometric progression in an MIA be matched to the sum of the terms of an arithmetic sequence in an AIA. We compare these special cases in Table 2.

Table 2

Special case of a 6 by 6 AIA		**Corresponding special case of a 6 by 6 MIA**	
1a	The first row and column are arithmetic progressions	1b	The first row and column are geometric progressions
2a	Successive rows of this array are translations of the first row sequence	2b	Successive rows of this array are magnifications of the first row sequence
3a	$a_1 = 0$	3b	$A_1 = 1$
4a	The arithmetic mean, m, of a and b satisfies the condition: $m+m=a+b$ or $m = \frac{a+b}{2}$	4b	The geometric mean, M, of A and B satisfies the condition: $M \cdot M = A \cdot B$ or $M = \sqrt{A \cdot B}$
5a	$a_1+a_2+a_3+a_4+a_5+a_6 = 6 \cdot \frac{(a_1+a_6)}{2}$ or $3 \cdot (a_1+a_6)$ (the sum of 6 successive terms of an arithmetic progression)	5b	$A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6 = \sqrt{A_1 \cdot A_6}$ or $(A_1 \cdot A_6)^3$ (the product of 6 successive terms of a geometric progression).[5]
6a	The sum of any six numbers of the array, no two lying in the same row or column equals six times the arithmetic mean of the numbers in the upper left and lower right corners of the array.	5c	The product of any six numbers of the array, no two lying in the same row or column equals the sixth power of the geometric mean of the numbers in the upper left and lower right corners of the array.[6]

Example 5

Use the result in cell 6b of Table 2 to calculate the product of any 6 numbers in the array of Figure 8, no two of which lie in the same row or column.

Solution

In our previous encounter with the Figure 8 array, we were not in a position to benefit from the fact that the first row and the first column were both geometric progressions: (1, 2, 4, 8, 16, 32 and 1, 3, 9, 27, 81, 243). Using the result in cell 6b of Table 2, we are now able to calculate the invariant product in an easier manner. The product of any six numbers of the array, no two of which lie in the same row or column equals

$$\left(\sqrt{1 \cdot 7776}\right)^6 = 7776^3 = 470\ 184\ 984\ 576$$

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5. A proof of the formula in entry 5b of Table 2 can be found at http://en.wikipedia.org/wiki/Geometric_progression. Applying duality to Gauss' trick for producing the formula in 5a is more satisfying.

6. To prove entry 6b of Table 2 the reader could start with the general-case invariant product found in Table 1, and arrive at the special-case result by applying duality to the given proof of entry 6a and employing entry 5b.

Final remarks

Our paper on additively and multiplicatively invariant arrays provides a good opportunity to re-examine two comparable mathematical structures that are familiar to secondary school students. The topic lends itself to reaching new conclusions, by exploiting a rich duality to both translate features of one structure into the "language" of the other as well as to produce full proofs of our discoveries. The unit focuses on comparable features of addition and multiplication, and exploits a relatively unknown connection between arithmetic and geometric progressions.

The subject we have introduced invites further exploration. Suggestions for additional study include:

1. a generalisation of AIA and MIA to 3 and 4 dimensional arrays (including the "special" cases) as well as to larger-sized square arrays;
2. an examination of the effect on the invariant sum/product caused by interchanging rows or columns or all the rows with all the columns of a given AIA or MIA.

Nomenclature adopted in this paper, as well as similar notions of duality appear in a previous *ASMJ* article (Berenson, 2002) dealing with invariance, both additive and multiplicative, in the context of third-order magic squares. Study of that article would undoubtedly reinforce ideas developed in this paper.

Reference

Berenson, L. (2002). Something different on magic squares. *Australian Senior Mathematics Journal, 16* (1), 43–55.